Phys 410
Fall 2014
Lecture #18 Summary
30 October, 2014

Up to this point we have considered Newtonian dynamics and Lagrangian dynamics. Now we consider Hamiltonian dynamics. The Lagrangian is written in terms of n generalized coordinates and their time derivatives. This set of parameters constitutes a $2n-$ dimensional **state space**. The Hamiltonian is written in terms of the generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. This set of $2n$ parameters constitutes **phase space**.

Recall from Lecture 15 that the Hamiltonian was derived to be $\mathcal{H} = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$.

One can solve the n canonical momentum equations for $\dot{q}_i$ in terms of the coordinates q_i and momenta p_i to arrive at $\dot{q}_i = \dot{q}_i(q_i, p_i)$. With this, one can express the Hamiltonian in terms of coordinates and momenta alone $\mathcal{H}(q,p)$, essentially employing a Legendre transformation to move from $(q_i,\,\dot{q}_i)$ to (q_i, p_i) as the independent variables. Taking the derivative of the Hamiltonian with respect to q_i and p_i, one finds Hamilton's equations: $\dot{q}_i = \partial\mathcal{H}/\partial p_i$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i$, $i = 1, \dots, n$. This is a set of $2n$ first-order differential equations, as opposed to the set of n second-order differential equations one gets from Lagrange's equations.

The Hamiltonian dynamics formulation is useful for quantum mechanics and for classical statistical mechanics. As a way of solving classical mechanics problems it has few advantages over Lagrangian dynamics.

We considered the Hamiltonian description of a particle moving in one dimension under the influence of a conservative force and showed that Hamilton's equations can be used to reproduce Newton's second law of motion. The procedure of utilizing the Hamiltonian method is: (1) choose the generalized coordinates q_i, (2) write down $T, U,$ and $\mathcal{L}$ in terms of the coordinates and their time-derivatives, (3) compute the conjugate momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, (4) express the $\dot{q}_i$ in terms of $q_i\ and\ p_i$, (5) compute the Hamiltonian $\mathcal{H}$, and (6) write out and solve Hamilton's equations. We then used the Hamilton method to solve for the equations of motion of the modified Atwood Machine, shown in Problem 13.23 of Taylor (pages 553-554).

The generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, constitute a set of $2n$ quantities that span **phase space**. The instantaneous state of the entire system is summarized as a single mathematical point in this phase space. Call this point

$\vec{z} = (\vec{q}, \vec{p})$, where $\vec{q} = (q_1, \ldots q_n)$ is an ordered list of the n generalized coordinates, and $\vec{p} = (p_1, \ldots p_n)$ is the list of n conjugate momenta. Hamiltonian's equations describe how this point moves in phase space – in other words it describes the trajectory of the phase point. This is a deterministic equation for the evolution of the phase point. It shows that two trajectories that arise from two different initial conditions can never cross, because otherwise there would be two different trajectories arising from the same equation with the same instantaneous value of $\vec{z}$, contrary to the deterministic nature of the phase point evolution equation.

We considered the $2n = 2$ –dimensional phase space of a $n = 1$ one-dimensional harmonic oscillator. The trajectory of the phase point is an ellipse in the (x, p) phase plane.